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Axisymmetric free vibration of closed thin spherical nano-shell

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Abstract: This work investigates the free axisymmetric vibrations of a closed spherical nano-shell using the Eringen nonlocal elasticity theory. The motion equations are properly formulated considering the hypotheses of thin shells and the solution is obtained using the classical separation of variables method. The effect of the nonlocal parameter on the natural frequencies and modal shapes are discussed in comparison to their local counterparts. This study could be useful in biomedical and bioengineering applications as well as in other fields related with the nanotechnology.

Keywords*:* Nonlocal elasticity, Sphere vibration, Nanotechnology, Natural frequencies, Modal shapes

1. Introduction

Modern technological applications involve the use of systems which can be devised as micro or nanostructures. From the discovery of fullerenes [1] and Carbon Nanotubes (CNTs) [2], these applications have experienced an exponential growth, mainly in micro-or nano-electromechanical (MEMS or NEMS) devices [3], nanomachines [4–7], as well as in biotechnology and biomedical fields [8].

A main characteristic of these nanostructures is that their dimensions become comparable to the size of their material microstructure or the molecular distances, thus the size effects are significant regarding their mechanical behavior. These systems could be analysed using Molecular Dynamics [9–12]. Since the atomic and molecular approaches require a great computational effort, simplified models are useful for analysing the mechanical behavior of such devices. However, classical continuum mechanics cannot predict the size effect, because the constitutive equations derived from this framework lack an internal length, characterizing the underlying microstructure, i.e. it is a scale-free theory.

Despite some sporadic efforts in the 19th century due to Cauchy and Voigt, and in the first half of the 20th century through the work of the Cosserat brothers to capture the effects of microstructure using the continuum equations of elasticity with additional higher-order derivatives, the major revival took place in the 1960s. From this time are the works of Mindlin and Tiersten [13], Kröner [14], Toupin [15,16], Green and Rivlin [17], Mindlin [18,19] and Mindlin and Eshel [20]. However, these formulations were excessively complex with too many parameters and equations. For instance, the more general form of the constitutive tensors proposed by Mindlin [18] includes 903 independent elastic constants that can be reduced to 18 for the isotropic case.

More recently, Eringen derived, from his earlier integral nonlocal theories [21], a simple stress-gradient formulation which contains a length scale parameter. In the early 1990s, Aifantis and coworkers suggested to extend the linear elastic constitutive relations with the Laplacian of the strain through a length scale parameter again [22–24]. Askes and Gitman [25] shown that an unification of both Eringen and Aifantis theories is possible. An overview on the historical development of theses theories, as well as its meaning and implementation can be found in the paper by Askes and Aifantis [26].

Among the size-dependent continuum theories, the theory of nonlocal continuum mechanics initiated by Eringen and coworkers [27,28,21] has been widely used to analyse many problems, such as wave propagation, dislocation, and crack singularities. From the pioneer work of Peddieson et al. [29], this theory has been also used to solve problems involving nanostructures. Thus, the Eringen nonlocal theory of elasticity has been used to address the behavior of beams [30–36], beams under rotation [37–40], rods [41–46], plates [47–49], cylindrical shells [50–52], conical shells [53,11,12], rings [54,55] and particles [56], as well as carbon nanotubes (CNTS) [57–64].

On the other hand, the dynamics of the closed spherical shells (fluid-filled or empty) is a problem of technological importance in some modern industrial, biomedical, biological and other applications [65,66]. There exists an extensive bibliography dedicated to the analysis of buckling and vibrations of spherical shells from the point of view of the classical elasticity theory [67–70]. The classical continuum framework has been also applied to the case of nanospheres [71,72]. In various modern biomedical and biological applications, spherical membrane structure (fluid-filled or empty) can be used to model some micro/nanosized components, which are used as targeted drug delivery systems [73], biological cells [74], and some kind of viruses [75,76].

The above cited works rest on the concepts of classical continuum mechanics. In the authors knowledge, the only analyses concerning spherical shells which take into account size effects using nonlocal continuum theories is due to Ghavanloo and Fazelzadeh [77], who presented a study on the radial vibrations of a closed spherical shell using the Eringen nonlocal elasticity theory, and the same authors [66], who studied the coupled axisymmetric vibrations of fluid-filled closed spherical membrane shell using the same nonlocal approach. However, for the case of axisymmetric vibrations, its formulation of the problem is not properly addressed because some issues concerning the handling of several differential operators in spherical coordinates are omitted and the analysis should be revisited.

In this paper we present a detailed study of free axisymmetric vibrations of a closed spherical nano-shell using the Eringen nonlocal elasticity theory. The hypothesis of thin shells have been taken into account and then bending moments, shear efforts and radial normal stresses were neglected. The solution method proposed follows the procedure used by Baker [68] for the elastic local case. The effect of the nonlocal parameter on the results, i.e. natural frequencies and mode shapes, are discussed.

2. General equations of Eringen elasticity theory

The Eringen formulation [28,78,79] states that the nonlocal stress-tensor $\mathbf{t}$ at any point $\mathbf{x}$ in a body can be expressed as

$$\mathbf{t}(\mathbf{x}) = \int_{\Omega} \alpha(|\mathbf{x}' - \mathbf{x}|, \gamma)\boldsymbol{\sigma}(\mathbf{x})d\Omega(\mathbf{x}') \tag{1}$$

where $\boldsymbol{\sigma}(\mathbf{x})$ is the classical local stress tensor at point $\mathbf{x}$, which is related to the linear strain tensor $\boldsymbol{\varepsilon}$ by the conventional constitutive relations for a Hookean material

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \tag{2}$$

where $\mathbf{C}$ is the fourth-order elasticity tensor and $\boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla(\mathbf{u}^T)) \tag{3}$$

$\mathbf{u}$ being the displacement vector.

Eq. (1) represents the weighted average of the contributions to the stress field at a point $\mathbf{x}$ of the strain field of all points in the body in the neighborhood of $\mathbf{x}$, the size of which is related to the nonlocal modulus $\alpha(|\mathbf{x}' - \mathbf{x}|, \gamma)$. Here, $|\mathbf{x}' - \mathbf{x}|$ is the Euclidean distance and γ is a material constant given by $\gamma = e_0 a/l$, that depends on internal and external characteristic lengths (a and l, respectively) trough an adjusting constant e_0, dependent on each material.

Both Eqs. (1) and (2) define the considered nonlocal constitutive behavior of a Hookean solid. For a class of physically admissible kernel $\alpha(|\mathbf{x}' - \mathbf{x}|, \gamma)$, Eringen [21] showed that the nonlocal constitutive equations given by the integral formulation could be replaced by gradients. Thus, Eq. (1) can be written in an equivalent differential form as

$$(1 - \kappa^2\nabla^2)\mathbf{t} = \boldsymbol{\sigma} \tag{4}$$

$\kappa = e_0 a$ being the length scale which takes into account the size effect on the response of nanostructures.

The balance of linear momentum results in the following equation of motion

$$\nabla \cdot \mathbf{t} + \mathbf{f} = \rho\ddot{\mathbf{u}} \tag{5}$$

where $\mathbf{f}$ represents the external body forces vector, and after using Eq. (4)

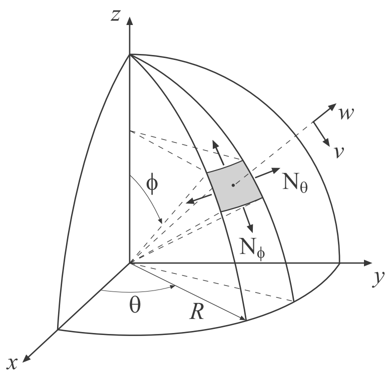

Fig. 1. Differential element of shell, membrane forces and symmetric displacements in spherical coordinates.

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = (1 - \kappa^2\nabla^2)\rho\ddot{\mathbf{u}} \tag{6}$$

Note that the displacement field of a nonlocal solid subject to an external body force field $\mathbf{f}$ and an inertial body force $-\rho\ddot{\mathbf{u}}$ is the same as that of a classical solid subject to the same external force $\mathbf{f}$ and an inertial body force $-(1 - \kappa^2\nabla^2)\rho\ddot{\mathbf{u}}$.

Considering that the material is isotropic, the equations of motion can be obtained in terms of the displacements

$$(1 - \kappa^2\nabla^2)\rho\ddot{\mathbf{u}} = (\lambda_L + G)\nabla(\nabla \cdot \mathbf{u}) + G\nabla^2\mathbf{u} + \mathbf{f} \tag{7}$$

λ_L and G being the Lamé constants. The above relation constitutes the Navier equation of motion for nonlocal solids, which must be solved with the appropriate initial and boundary conditions applicable in each case.

3. Axisymmetric motion for the nonlocal spherical shell

The above theory will be used to study the axisymmetric free vibration of a nonlocal closed thin spherical shell.

3.1. Problem formulation

We assume that the deformations are small enough for linear equations to adequately describe the motion. The thickness h of the shell is thin enough that bending moments, shear forces and radial normal stresses can be neglected. The sign convention for radial $w(\phi,t)$ and meridional $v(\phi,t)$ displacements, and for stress resultants in meridional $N_\phi(\phi,t)$ and circumferential $N_\theta(\phi,t)$ directions, are defined in Fig. 1. The governing equations of motion for the nonlocal spherical shell can be directly derived from the classical formulation for local elasticity [68], as a result of the aforementioned analogy between the nonlocal solid subjected to an inertial body force field $-\rho\ddot{\mathbf{u}}$ and the equivalent local solid subjected to an inertial body force field $-(1 - \kappa^2\nabla^2)\rho\ddot{\mathbf{u}}$. Neglecting the external body force $\mathbf{f}$, in order to consider free vibrations, the corresponding equations may be expressed as

$$\begin{aligned}&-(\widehat{N}_\phi + \widehat{N}_\theta)\\ &= \rho Rh\left(\frac{\partial^2 w}{\partial t^2} - \frac{\kappa^2}{R^2}\left(\frac{\partial^4 w}{\partial t^2\partial\phi^2} + \cot\phi\frac{\partial^3 w}{\partial t^2\partial\phi} - 2\frac{\partial^2 w}{\partial t^2} - 2\frac{\partial^3 v}{\partial t^2\partial\phi} - 2\cot\phi\frac{\partial^2 v}{\partial t^2}\right)\right)\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{\partial\widehat{N}_\phi}{\partial\phi} + \cot\phi(\widehat{N}_\phi - \widehat{N}_\theta)\\ &= \rho Rh\left(\frac{\partial^2 v}{\partial t^2} - \frac{\kappa^2}{R^2}\left(\frac{\partial^4 v}{\partial t^2\partial\phi^2} + \cot\phi\frac{\partial^3 v}{\partial t^2\partial\phi} - \frac{1}{\sin^2\phi}\frac{\partial^2 v}{\partial t^2} + 2\frac{\partial^3 w}{\partial t^2\partial\phi}\right)\right)\end{aligned} \tag{9}$$

where the proper differential operators in spherical coordinates have been used. R, ρ, E and ν are the sphere radius, mass density, Young modulus and Poisson ratio respectively. $\widehat{N}_\phi$ and $\widehat{N}_\theta$ are the stress resultants in the equivalent local problem, that can be expressed in terms of the displacements as can be found, for instance, in Soedel [70]

$$\widehat{N}_\phi = \frac{Eh}{R(1-\nu^2)}\left(\frac{\partial v}{\partial \phi} + w + \nu(\cot\phi\, v + w)\right) \tag{10}$$

$$\widehat{N}_\theta = \frac{Eh}{R(1-\nu^2)}\left(\cot\phi\, v + w + \nu\left(\frac{\partial v}{\partial \phi} + w\right)\right) \tag{11}$$

Considering the following nondimensional quantities

$$\bar{v} = \frac{v}{R};\quad \bar{w} = \frac{w}{R};\quad \tau = \frac{c_s t}{R};\quad c_s = \sqrt{\frac{E}{\rho(1-\nu^2)}};\quad \mu = \frac{\kappa}{R} \tag{12}$$

the nondimensional equations of motion may be finally written in terms of displacements as

$$-(1+\nu)\left(\frac{\partial\bar{v}}{\partial\phi} + \cot\phi\,\bar{v} + 2\bar{w}\right) = \frac{\partial^2\bar{w}}{\partial\tau^2} - \mu^2\left(\frac{\partial^4\bar{w}}{\partial\tau^2\phi^2} + \cot\phi\frac{\partial^3\bar{w}}{\partial\tau^2\phi} - 2\frac{\partial^2\bar{w}}{\partial\tau^2} - 2\frac{\partial^3\bar{v}}{\partial\tau^2\partial\phi} - 2\cot\phi\frac{\partial^2\bar{v}}{\partial\tau^2}\right) \tag{13}$$

$$\frac{\partial^2\bar{v}}{\partial\phi^2} + \frac{\partial\bar{v}}{\partial\phi}\cot\phi - (\nu + \cot^2\phi)\bar{v} + (1+\nu)\frac{\partial\bar{w}}{\partial\phi} = \frac{\partial^2\bar{v}}{\partial\tau^2} - \mu^2\left(\frac{\partial^4\bar{v}}{\partial\tau^2\phi^2} + \cot\phi\frac{\partial^3\bar{v}}{\partial\tau^2\phi} - \csc^2\phi\frac{\partial^2\bar{v}}{\partial\tau^2} + 2\frac{\partial^3\bar{w}}{\partial\tau^2\phi}\right) \tag{14}$$

These equations are identical to the local formulation if the nonlocal parameter μ is set to zero. Note that, for $\mu > 0$, Eqs. (13) and (14) are not coincident with those proposed by Fazelzadeh and Ghavanloo [66].

Following Baker [68] we will assume that the shell is initially at rest in a deformed shape, according to the following initial conditions

$$\bar{w}(\phi,0) = f(\phi);\quad \left.\frac{\partial\bar{w}}{\partial\tau}\right|_{\tau=0} = 0 \tag{15}$$

$$\bar{v}(\phi,0) = g(\phi);\quad \left.\frac{\partial\bar{v}}{\partial\tau}\right|_{\tau=0} = 0 \tag{16}$$

Eqs. (13) and (14), subject to the initial conditions (15) and (16), determine the free vibrational movement of the nonlocal spherical shell.

3.2. Solving method

The differential equations to be solved are linear and homogeneous. Thus, in order to obtain the natural frequencies and modal shapes of the vibration of the shell, the method of separation of variables will be used. Let us assume

$$\bar{w}(\phi,\tau) = \sum_n W_n(\phi)S_n(\tau) \tag{17}$$

$$\bar{v}(\phi,\tau) = \sum_n V_n(\phi)Q_n(\tau) \tag{18}$$

Then, Eqs. (13) and (14) become

$$-2(1+\nu)W_nS_n - (1+\nu)(V_n' + \cot\phi V_n)Q_n = \left(W_n - \mu^2(W_n'' + \cot\phi W_n\prime - 2W_n)\right)\frac{d^2S_n}{d\tau^2} + 2\mu^2(V_n' + \cot\phi V_n)\frac{d^2Q_n}{d\tau^2} \tag{19}$$

$$(1+\nu)W_n'S_n + \left(V_n'' + \cot\phi V_n' - (\nu + \cot^2\phi)V_n\right)Q_n = -2\mu^2W_n'\frac{d^2S_n}{d\tau^2} + \left(V_n - \mu^2(V_n'' + \cot\phi V_n' - \csc^2\phi V_n)\right)\frac{d^2Q_n}{d\tau^2} \tag{20}$$

or, dividing Eq. (19) by $\left(W_n - \mu^2(W_n'' + \cot\phi W_n' - 2W_n)\right)$ and Eq. (20) by $\left(V_n - \mu^2(V_n'' + \cot\phi V_n' - \csc^2\phi V_n)\right)$

$$-\lambda_1S_n - (1+\nu)\lambda_2Q_n = \frac{d^2S_n}{d\tau^2} + 2\mu^2\lambda_2\frac{d^2Q_n}{d\tau^2} \tag{21}$$

$$(1+\nu)\lambda_3S_n + \lambda_4Q_n = -2\mu^2\lambda_3\frac{d^2S_n}{d\tau^2} + \frac{d^2Q_n}{d\tau^2} \tag{22}$$

with

$$\lambda_1 = \frac{2(1+\nu)W_n}{W_n - \mu^2(W_n'' + \cot\phi W_n' - 2W_n)} \tag{23}$$

$$\lambda_2 = \frac{V_n' + \cot\phi V_n}{W_n - \mu^2(W_n'' + \cot\phi W_n' - 2W_n)} \tag{24}$$

$$\lambda_3 = \frac{W_n'}{V - \mu^2(V_n'' + \cot\phi V_n' - \csc^2\phi V_n)} \tag{25}$$

$$\lambda_4 = \frac{V_n'' + \cot\phi V_n' - (\nu + \cot^2\phi)V_n}{V_n - \mu^2(V_n'' + \cot\phi V_n' - \csc^2\phi V_n)} \tag{26}$$

For Eqs. (21) and (22) to be separable, λ_i must be constant.

We can now determine the constants λ_1 to λ_4 and the functional forms of W_n and V_n.

3.3. Determination of λ_i and functional forms W_n and V_n

For convenience we let $x = \cos\phi$. Then Eq. (23) leads to the Legendre differential equation [80]

$$(1-x^2)\frac{d^2W_n}{dx^2} - 2x\frac{dW_n}{dx} + \frac{2(1+\nu-\mu^2\lambda_1)-\lambda_1}{\mu^2\lambda_1}W_n = 0 \tag{27}$$

For radial displacements W_n to be bounded over the entire sphere, the following condition is required

$$\frac{2(1+\nu-\mu^2\lambda_1)-\lambda_1}{\mu^2\lambda_1} = n(n+1) \tag{28}$$

or

$$\lambda_1 = \frac{2(1+\nu)}{1+\mu^2(2+n(n+1))} \tag{29}$$

and the solution for W_n is given by the Legendre polynomials

$$W_n(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n} \tag{30}$$

with n integer greater or equal than zero.

In a similar way, Eq. (26) leads to a general Legendre differential equation

$$(1-x^2)\frac{d^2V_n}{dx^2} - 2x\frac{dV_n}{dx} + \left(\frac{1-\nu-\lambda_4}{1+\mu^2\lambda_4} - \frac{1}{1-x^2}\right)V_n = 0 \tag{31}$$

For meridional displacements V_n being finite and single-valued over the entire sphere, the following condition is required

$$\frac{1-\nu-\lambda_4}{1+\mu^2\lambda_4} = n(n+1) \tag{32}$$

or

$$\lambda_4 = \frac{1-\nu-n(n+1)}{1+\mu^2n(n+1)} \tag{33}$$

and the solution for V_n is given by the associated Legendre Polynomials of the first kind (Condon–Shortley phase ommited, see Arfken et al. [81])

$$V_n(x) = P_n^1(x) = \frac{(1-x^2)^{1/2}}{2^n n!}\frac{d^{n+1}(x^2-1)^n}{dx^{n+1}} \tag{34}$$

with n integer greater than zero.

To obtain λ_2 we use the following relation between Legendre polynomials and associated Legendre polynomials of the first kind

$$P_n^1(x) = (1-x^2)^{1/2}\frac{dP_n(x)}{dx} \tag{35}$$

which is equivalent to

$$V_n = -W_n' \tag{36}$$

Substituting Eq. (36) and its derivative into Eq. (24) to eliminate V_n and V_n', we get again the Legendre differential equation

$$(1-x^2)\frac{d^2W_n}{dx^2} - 2x\frac{dW_n}{dx} - \frac{\lambda_2(1+2\mu^2)}{\mu^2\lambda_2 - 1}W_n = 0 \tag{37}$$

For W_n to be a bounded solution of this equation, the following condition is required

$$-\frac{\lambda_2(1+2\mu^2)}{\mu^2\lambda_2 - 1} = n(n+1) \tag{38}$$

or

$$\lambda_2 = \frac{n(n+1)}{1+\mu^2(2+n(n+1))} \tag{39}$$

Now substituting Eq. (36) in Eq. (25) to eliminate W_n', we obtain a general Legendre differential equation

$$(1-x^2)\frac{d^2V_n}{dx^2} - 2x\frac{dV_n}{dx} + \left(\frac{-1-\lambda_3}{\mu^2\lambda_3} - \frac{1}{1-x^2}\right)V_n = 0 \tag{40}$$

For V_n to be a finite and single-valued solution of this equation, the following condition is required

$$\frac{-1-\lambda_3}{\mu^2\lambda_3} = n(n+1) \tag{41}$$

or

$$\lambda_3 = -\frac{1}{1+\mu^2 n(n+1)} \tag{42}$$

3.4. Solution for the time-dependent functions S_n and Q_n

By suitable combinations of Eqs. (21) and (22), we can transform them in two uncoupled biquadratic differential equations

$$\frac{d^4S_n}{d\tau^4} + \xi_n\frac{d^2S_n}{d\tau^2} + \gamma_n S_n = 0 \tag{43}$$

$$\frac{d^4Q_n}{d\tau^4} + \xi_n\frac{d^2Q_n}{d\tau^2} + \gamma_n Q_n = 0 \tag{44}$$

Note that both equations have the same coefficients

$$\xi_n = \frac{(1+n(n+1))(1+\mu^2(-2+n(n+1))) - \nu(-3+\mu^2(-2+n(n+1)))}{1+\mu^2[2(1+n(n+1))+\mu^2 n(n+1)(-2+n(n+1))]} \tag{45}$$

$$\gamma_n = \frac{(1-\nu^2)(-2+n(n+1))}{1+\mu^2[2(1+n(n+1))+\mu^2 n(n+1)(-2+n(n+1))]} \tag{46}$$

The solution of these linear differential equations can be written as

$$S_n(\tau) = A_n^S\cos(a_n\tau) + B_n^S\sin(a_n\tau) + C_n^S\cos(b_n\tau) + D_n^S\sin(b_n\tau) \tag{47}$$

$$\begin{aligned} Q_n(\tau) = {} & A_n^Q\cos(a_n\tau) + B_n^Q\sin(a_n\tau) \\ & + C_n^Q\cos(b_n\tau) + D_n^Q\sin(b_n\tau) \end{aligned} \tag{48}$$

where the natural frequencies are roots of the corresponding characteristic equation of the differential problem given by Eqs. (43) and (44)

$$a_n = \left(\frac{\xi_n + \sqrt{\xi_n^2 - 4\gamma_n}}{2}\right)^{1/2} \tag{49}$$

$$b_n = \left(\frac{\xi_n - \sqrt{\xi_n^2 - 4\gamma_n}}{2}\right)^{1/2} \tag{50}$$

As in the local elasticity case, there are two natural frequencies for each value of n: the upper branch a_n and the lower branch b_n. The eight unknown amplitudes in Eqs. (47) and (48) are obtained, applying the corresponding initial conditions, in terms of the initial values of the functions S_n and Q_n and of their first, second and third derivatives

$$S_n(0) = S_n^0; \quad Q_n(0) = Q_n^0 \tag{51}$$

$$\left.\frac{dS_n}{d\tau}\right|_{\tau=0} = \left.\frac{dQ_n}{d\tau}\right|_{\tau=0} = 0 \tag{52}$$

$$\left.\frac{d^2Q_n}{d\tau^2}\right|_{\tau=0} = \ddot{Q}_n^0 = p_n S_n^0 + q_n Q_n^0 \tag{53}$$

$$\left.\frac{d^2S_n}{d\tau^2}\right|_{\tau=0} = \ddot{S}_n^0 = r_n S_n^0 + s_n Q_n^0 \tag{54}$$

and

$$\left.\frac{d^3S_n}{d\tau^3}\right|_{\tau=0} = \left.\frac{d^3Q_n}{d\tau^3}\right|_{\tau=0} = 0 \tag{55}$$

with

$$p_n = \frac{(1+\nu-2\ \mu^2\lambda_1)\lambda_3}{1+4\ \mu^4\lambda_2\lambda_3} \tag{56}$$

$$q_n = \frac{-2\ \mu^2(1+\nu)\lambda_2\lambda_3 + \lambda_4}{1+4\ \mu^4\lambda_2\lambda_3} \tag{57}$$

$$r_n = -\frac{\lambda_1 + 2\ \mu^2(1+\nu)\lambda_2\lambda_3}{1+4\ \mu^4\lambda_2\lambda_3} \tag{58}$$

$$s_n = -\frac{\lambda_2(1+\nu+2\ \mu^2\lambda_4)}{1+4\ \mu^4\ \lambda_2\lambda_3} \tag{59}$$

Initial conditions (51) and (52) correspond to Eqs. (15) and (16). Initial conditions (53) and (54) are obtained combining Eqs. (21) and (22), whereas initial conditions (55) are obtained by derivation of Eqs. (21) and (22). The solution of the algebraic system of equations given by the eight initial conditions, Eqs. (51)–(55), leads to

$$B_n^S = D_n^S = B_n^Q = D_n^Q = 0 \tag{60}$$

$$A_n^S = S_n^0\frac{\ddot{S}_n^0/S_n^0 + b_n^2}{b_n^2 - a_n^2} \tag{61}$$

$$C_n^S = S_n^0 \frac{\ddot{S}_n^0/S_n^0 + a_n^2}{a_n^2 - b_n^2} \tag{62}$$

$$A_n^Q = Q_n^0 \frac{\ddot{Q}_n^0/Q_n^0 + b_n^2}{b_n^2 - a_n^2} \tag{63}$$

$$C_n^Q = Q_n^0 \frac{\ddot{Q}_n^0/Q_n^0 + a_n^2}{a_n^2 - b_n^2} \tag{64}$$

Taking into account conditions (53) and (54), and the relation $a_n^2 + b_n^2 = \xi_n$, the final expression for the time-dependent functions are

$$S_n(\tau) = S_n^0 \left[\frac{\beta_n - a_n^2}{b_n^2 - a_n^2} \cos(a_n\tau) + \frac{\beta_n - b_n^2}{a_n^2 - b_n^2} \cos(b_n\tau) \right] \tag{65}$$

$$Q_n(\tau) = Q_n^0 \left[\frac{\alpha_n - a_n^2}{b_n^2 - a_n^2} \cos(a_n\tau) + \frac{\alpha_n - b_n^2}{a_n^2 - b_n^2} \cos(b_n\tau) \right] \tag{66}$$

where

$$\beta_n = \frac{\ddot{S}_n^0}{S_n^0} + \xi_n \tag{67}$$

$$\alpha_n = \frac{\ddot{Q}_n^0}{Q_n^0} + \xi_n \tag{68}$$

The initial values S_n^0 and Q_n^0 can be obtained following the procedure proposed by Baker [68]. Considering Eqs. (17) and (18) at $t = 0$

$$\bar{w}(x, 0) = \sum_{n=0}^{\infty} P_n(x) S_n^0 \tag{69}$$

$$\bar{v}(x, 0) = \sum_{n=1}^{\infty} P_n^1(x) Q_n^0 \tag{70}$$

multiplying the above expressions by $P_m(x)$ and $P_m^1(x)$ respectively, integrating over $0 \leqslant \phi \leqslant \pi$ and using the orthogonal properties of the Legendre Polynomials, we get

$$S_n^0 = -\frac{2n+1}{2} \int_1^{-1} P_n(x) \bar{w}(x, 0) dx \tag{71}$$

$$Q_n^0 = -\frac{(2n+1)(n-1)!}{2(n+1)!} \int_1^{-1} P_n^1(x) \bar{v}(x, 0) dx \tag{72}$$

3.5. Modal shapes

We can find the modal shape vibrating with upper branch frequency a_n by setting to zero the amplitudes corresponding to the lower branch frequency in Eqs. (65) and (66)

$$\beta_n - b_n^2 = 0; \quad \alpha_n - b_n^2 = 0 \tag{73}$$

therefore

$$\beta_n = \alpha_n \tag{74}$$

or

$$\frac{\ddot{S}_n^0}{S_n^0} = \frac{\ddot{Q}_n^0}{Q_n^0} \tag{75}$$

Combining the last condition with Eqs. (53) and (54) we get

$$p_n \left(\frac{S_n^0}{Q_n^0} \right)^2 + (q_n - r_n) \left(\frac{S_n^0}{Q_n^0} \right) - s_n = 0 \tag{76}$$

and solving this quadratic equation

$$\Psi_{a_n} = \left. \frac{S_n^0}{Q_n^0} \right|_{a_n} = \frac{r_n - q_n - \sqrt{(q_n - r_n)^2 + 4p_n s_n}}{2p_n} \tag{77}$$

where the negative sign before the radical is chosen to satisfy either of conditions (73). Finally, the temporal function corresponding to the frequency a_n is given by

$$S_n^{a_n}(\tau) = S_n^0\big|_{a_n} \cos(a_n\tau); \quad Q_n^{a_n}(\tau) = Q_n^0\big|_{a_n} \cos(a_n\tau) \tag{78}$$

Likewise, we can get the modal shape vibrating with lower frequency b_n by setting to zero the amplitudes corresponding to the upper branch frequencies in Eqs. (65) and (66)

$$\beta_n - a_n^2 = 0; \quad \alpha_n - a_n^2 = 0 \tag{79}$$

leading to

$$\Psi_{b_n} = \left. \frac{S_n^0}{Q_n^0} \right|_{b_n} = \frac{r_n - q_n + \sqrt{(q_n - r_n)^2 + 4p_n s_n}}{2p_n} \tag{80}$$

where the positive sign before the radical is chosen to satisfy either of conditions (79). Therefore, the temporal function corresponding to the frequency b_n is given by

$$S_n^{b_n}(\tau) = S_n^0\big|_{b_n} \cos(b_n\tau); \quad Q_n^{b_n}(\tau) = Q_n^0\big|_{b_n} \cos(b_n\tau) \tag{81}$$

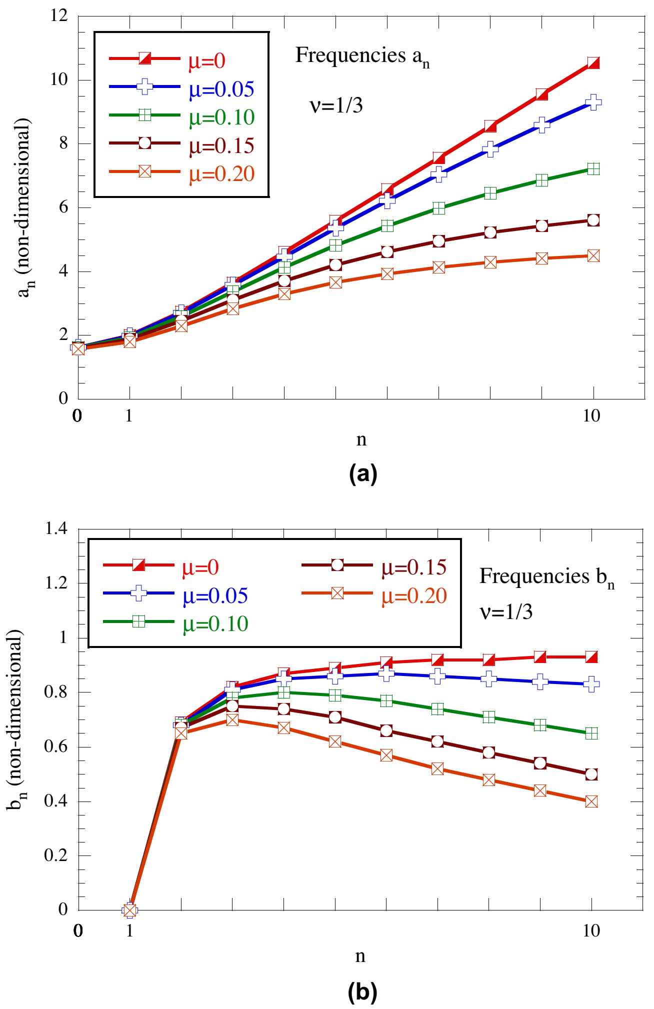

Fig. 2. Natural frequencies for different values of μ ($\nu = 1/3$). (a) Upper branch a_n. (b) Lower branch b_n.

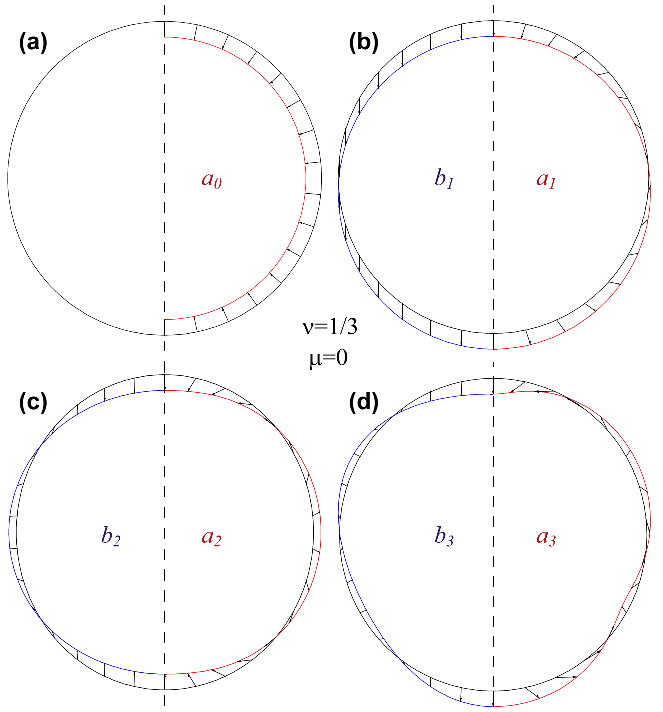

Fig. 3. Modal shapes for: (a) $n = 0$, (b) $n = 1$, (c) $n = 2$, and (d) $n = 3$; ($\nu = 1/3$, $\mu = 0$, $\epsilon = 0.1$). Black: undeformed configuration. Red: modal shape for a_n. Blue: modal shape for b_n. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The solution of Baker is recovered for $\mu = 0$, exceptuating the sign of the expression, which is consistent with the positive direction for the radial displacement taken in the present work.

4. Analysis of the influence of the nonlocal parameter

4.1. Frequencies

Eqs. (49) and (50) provide the natural frequencies of the free vibration of the shell. It is worth to highlight that the values calculated by Baker [68] are recovered by setting $\mu = 0$. As in the local elasticity case, b_0 leads to a spurious mode, and b_1 equals zero for every value of the nonlocal parameter thus forecasting a translational mode that will be confirmed afterwards. Fig. 2 shows the frequencies for n from 0 through 10, for different values of the nonlocal parameter. The frequencies of both upper and lower branches decrease with increasing values of μ. The translational mode b_1 is accountably independent of the nonlocal parameter, and the breathing mode a_0 is independent as well. The lower frequencies a_1, a_2, and b_2 are slightly affected by μ. The effect of the nonlocal parameter starts to play a major role for values of $n \geqslant 3$. Actually there are remarkable differences between local and nonlocal theories for large values of n, even for small values of μ.

4.2. Modal shapes

To draw the modal shapes, let us fix the initial radial displacement at the north pole to

$$w_n(0,0) = -\epsilon R \tag{82}$$

with $0 < \epsilon \ll 1$. Therefore

$$\bar{w}_n(0,0) = W_n|_{\phi=0} S_n^0 = W_n|_{x=1} S_n^0 = -\epsilon \tag{83}$$

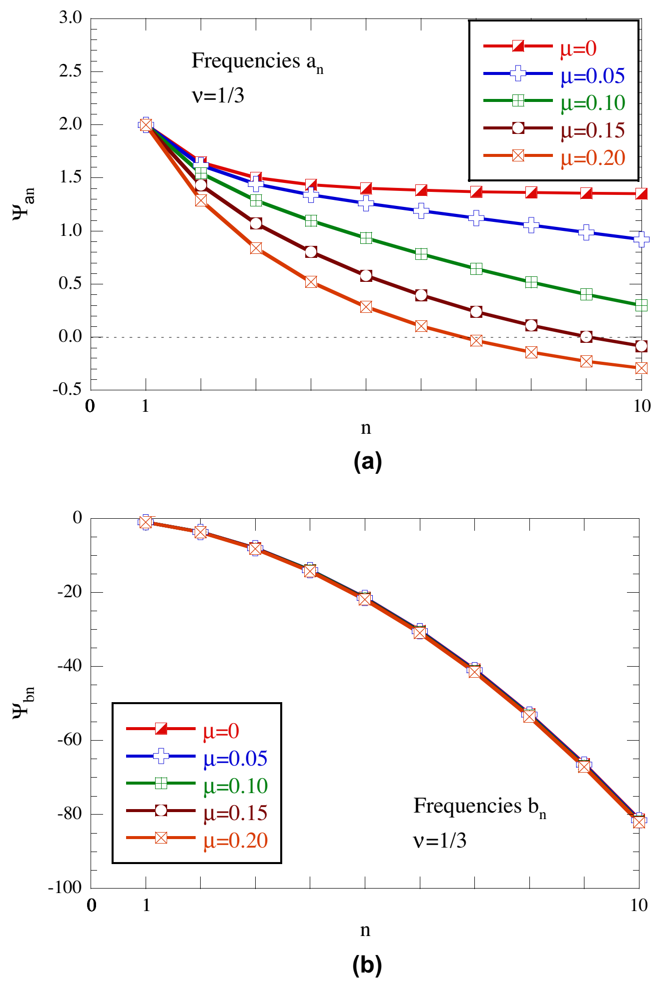

Fig. 4. Ratio $\phi = 0$ for different values of μ ($\nu = 1/3$). (a) Upper branch a_n. (b) Lower branch b_n.

Since $W_n = P_n$, condition (83) leads to $S_n^0 = -\epsilon$, and the initial amplitudes of the tangential displacement, for frequencies a_n and b_n, are given by

$$Q_n^0\Big|_{a_n} = -\frac{\epsilon}{\Psi_{a_n}}; \quad Q_n^0\Big|_{b_n} = -\frac{\epsilon}{\Psi_{b_n}} \tag{84}$$

Finally, the modal shapes can be written as

$$w_n(\phi,0) = -\epsilon R P_n(\cos\phi) \tag{85}$$

$$v_n(\phi,0) = -\frac{\epsilon}{\Psi_n} R P_n^1(\cos\phi) \tag{86}$$

with $\Psi_n = \Psi_{a_n}$ or $\Psi_n = \Psi_{b_n}$ depending on the branch of frequencies.

Fig. 3 shows, for $\mu = 0$, the modal shapes for n from 0 through 3. The solution given by Baker [68] is recovered, including the translational mode corresponding to b_1.

In order to evaluate the influence of the nonlocal parameter on the amplitude of the meridional displacement v, the factor Ψ_n in Eq. (86) is plotted in Fig. 4 for the upper and lower branches. It can be observed that Ψ_{an} decreases with n for every value of μ. Moreover, for $\mu > 0$, the Ψ_{a_n} curves intersect the horizontal axis. Since v increases as the absolute value of Ψ_{a_n} decreases, this may lead to large values of the meridional displacement for certain modal shapes, as can be seen in Fig. 5b.

Regarding the lower branch (Fig. 4b), Ψ_{b_n} is not affected by the nonlocal parameter. Thus, the nonlocal modal shapes are analogous to their local counterparts.

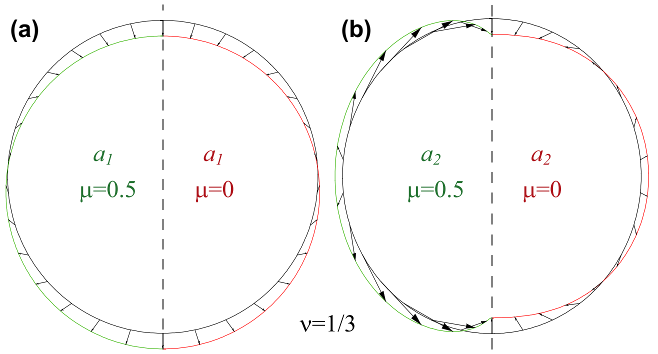

Fig. 5. Modal shapes a_n for: (a) n = 1 and (b) n = 2; ($\nu = 1/3$, $\epsilon = 0.1$). Black: undeformed configuration. Red: modal shape for $\mu = 0$. Green: modal shape for $\mu = 0.5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Conclusions

A detailed study of free axisymmetric vibrations of a closed spherical nano-shell using the Eringen nonlocal elasticity theory is presented. The hypotheses of thin shells are taken into account and then bending moments, shear efforts and radial normal stresses are neglected.

Using the classical variable separation method, the solution is assumed as a sum of products of both space-dependent and time-dependent functions, permitting to obtain the natural frequencies and modal shapes. As in the local elasticity case, there are two natural frequencies for each value of n, the upper branch and the lower branch, with a specific vibration mode for each frequency.

The following conclusions on the effect of nonlocal parameter in natural frequencies and modal shapes can be established:

- The frequencies of both upper and lower branches decrease with increasing values of the nonlocal parameter μ.
- The breathing mode a_0 is independent of the nonlocal parameter. The lower frequencies a_1, a_2, and b_2 are slightly affected by μ.
- Remarkable differences between local and nonlocal theories, even for small values of μ, appear for larger values of n ($n \geqslant 3$).
- The ratio of the radial to meridional displacements, corresponding to certain modal shapes associated to the upper branch, approaches zero for $\mu > 0$. This may lead to large values of the meridional displacement in the nonlocal case.
- The modal shapes corresponding to the lower branch frequencies are not affected by the nonlocal parameter, thus the nonlocal modal shapes are analogous to their local counterparts.

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